Corrections to
Introduction to Smooth Manifolds
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Changes or additions made in the past twelve months are dated.

(7/5/06) **Page 6, line 5:** Replace $\mathbb{R}^n$ by $\mathbb{R}^{n+1}$.

- **Page 6, lines 6 and 3 from the bottom:** Replace $U_i^+ \cap \mathbb{S}^n$ by U_i^+, and replace $U_i^- \cap \mathbb{S}^n$ by U_i^-.
- **Page 7, lines 1 and 2:** Replace $U_i^\pm \cap \mathbb{S}^n$ by $U_i^\pm$ (twice).
- **Page 9, last line of first paragraph:** Insert "is" before "the same."
- **Page 10, last line:** Replace $\widetilde{f}_n$ by $\widetilde{f}_k$.
- **Page 17, Example 1.14:** In the sixth line of this example, replace $\mathrm{Id}_{\mathbb{R}^n}$ by $\mathrm{Id}_{\mathbb{R}}$.
- **Page 18, second line below the heading:** After "throughout the book," insert "(except in the Appendix, which is designed to be read before the rest of the book)."
- **Page 19, line 6 from the bottom:** Replace A_U by $\mathcal{A}_U$.

(3/7/07) **Page 22, Figure 1.9:** Replace $\varphi(U_\alpha)$ by $\varphi_\alpha(U_\alpha)$ and $\varphi(U_\beta)$ by $\varphi_\beta(U_\beta)$.

- **Page 33, proof of Lemma 2.2, fourth line:** Replace $\varphi(V)$ by $\psi(V)$.

(8/24/06) **Page 38, Example 2.7(b), first line:** Insert "invertible" before "complex."

- **Page 41, line 2:** After the period, insert the sentence "It is called a *universal covering manifold* if $\widetilde{M}$ is simply connected."
- **Page 54, line 4:** should read $\overline{V}_k \cap V_j \neq \varnothing$ (instead of 0).
- **Page 57, Problem 2-1:** Change "representation" to "representations."

(7/19/06) **Page 62, Figure 3.1:** Change V_a to v_a (lowercase v).

(7/19/06) **Page 62, last paragraph, line 3:** change "Euclidean tangent vector" to "geometric tangent vector."

(3/27/06) **Page 67, line 4 from the bottom:** Delete redundant "the."

- **Page 74, line above the first displayed equation:** Insert "on some neighborhood of a" after "$\widetilde{f} \circ \iota = f$" and before the comma.
- **Page 74, line below the first displayed equation:** Replace $\mathbb{T}_a\mathbb{R}^n$ by $T_a\mathbb{R}^n$.
- **Page 89, line 9:** Change "preceding lemma" to "preceding proposition."

(7/19/06) **Page 91, lines 1 and 2:** Change "its values are determined locally" to "its action on a function is determined locally."

- **Page 93, two lines above Lemma 4.18:** Change $\mathcal{T}(M)$ to $\mathcal{T}(G)$.

(9/18/06) **Page 95, line 4 from the bottom, and page 96, line 1:** Change $(-\varepsilon, \varepsilon)$ to $(-\delta, \delta)$ (to avoid conflict with the evaluation map ε).

- **Page 96, proof of Corollary 4.21:** The following rephrasing might be clearer: "Let Y be a left-invariant rough vector field on a Lie group G, and let $V = Y_e$. The fact that Y is left-invariant implies that $Y = \widetilde{V}$, which is smooth."

(2/15/07) **Page 100, Proposition 4.26:** In the statement of part (a) and in the first line of the proof, replace $(\mathrm{Id}_G)*$ by $(\mathrm{Id}_G)_*$ (the asterisk should be set as a subscript).

- **Page 109, line 4 below the section heading:** Replace "each $p \in U$" by "each $p \in M$."

- **Page 115, just above the section heading:** Delete the sentence beginning "A remarkable theorem ...." [*The theorem actually proved by Wolf requires an extra consistency hypothesis in addition to parallelizability, so it does not apply to arbitrary compact, simply connected parallelizable manifolds. Similar misstatements of Wolf's theorem are common in the literature, so be careful whenever you see this theorem quoted. My apologies for perpetuating the misunderstanding.*]

- **Page 117, last displayed equation:** Replace $\mathcal{A}_i^j(q)$ by $A_i^j(q)$.

- **Page 117, last line:** Replace "trivializations of E" by "trivializations of E and E'."

- **Page 127, first line:** Replace V^* by V^{**}.

(7/20/06) **Page 129, statement of Proposition 6.5, first line:** Change "*manifold*" to "*n-manifold*."

- **Page 129, two lines above the last displayed equation:** Replace $\pi^{-1}(U \cap V)$ by $\pi^{-1}(U \cap \widetilde{U})$.

- **Page 129, line 3 from the bottom:** Replace "Proposition 5.3" by "Lemma 5.14."

- **Page 131, line 3:** Replace (ε_i) by (ε^i).

- **Page 131, line 5:** Replace "vector fields" by "covector fields."

- **Page 135, Figure 6.2:** The axis label on the right should be $x^2, \dots, x^n$.

(7/20/06) **Page 135, first line of text:** The domain of $f \circ \gamma$ should be J, not $\mathbb{R}$.

- **Page 143, eighth line after the section heading:** Change "Lemma 6.10" to "Proposition 6.10."

- **Page 151, Problem 6-2:** Replace the problem statement by the following:

  (a) If $F\colon M \longrightarrow N$ is a diffeomorphism, show that $F^*\colon T^*N \longrightarrow T^*M$ is a smooth bundle map.

  (b) Show that the assignment $M \mapsto T^*M$, $F \mapsto F^*$ defines a contravariant functor from SM_1 to VB, where SM_1 is the subcategory of SM whose objects are smooth manifolds, but whose morphisms are only diffeomorphisms.

- **Page 152, Problem 6-8(b):** Add the following hint: "[Hint: Use Proposition 5.16, noting that $C^\infty(M)$ can be naturally identified with the space of smooth sections of the trivial line bundle $M \times \mathbb{R}$.]"

- **Page 154, Problem 6-14:** Throughout the problem, change the index i to j, to avoid confusion with $i = \sqrt{-1}$ in $e^{2\pi i t}$ and $e^{2\pi i x^j}$.

(2/21/06) **Page 158, first displayed equation:** There is a missing factor of c in an exponent in the second member of the equation. The entire line should read

$$\left|e^{2\pi ick} - 1\right| = \left|e^{-2\pi icn_2}\left(e^{2\pi icn_1} - e^{2\pi icn_2}\right)\right| = \left|e^{2\pi icn_1} - e^{2\pi icn_2}\right| < \varepsilon.$$

- **Page 164, line 7:** The range of $\widetilde{R}$ should be $\mathbb{R}^{n-k}$, not $\mathbb{R}^k$.

- **Page 175, line below the first three displayed equations:** Replace the phrase "both φ and π are open maps" by "$\varphi(V)$ is the intersection of $\varphi(U)$ with an affine subspace $A \subset \mathbb{R}^n$ and is therefore open in A, and $\pi|_A$ is a homeomorphism."
- **Page 177, proof of Theorem 8.3:** In the second line of the proof, replace U by V (twice).
- **Page 186, line 10 from the bottom:** Change N to S (twice).
- **Page 189, third line after the subheading:** Change "a immersion" to "an immersion."
- **Page 196, Example 8.34, first line:** Add "$\mathrm{SL}(n, \mathbb{R})$" after the words "The set."
- **Page 209, Example 9.1(e), line 5:** Replace "any $v \in \mathbb{S}^n$" by "any $v \in \mathbb{S}^{n-1}$."
- **Page 217, line 4:** Replace $\lim_i(g_i \cdot q_i)$ by $\lim_i(g_i \cdot p_i)$.
- **Page 217, line 5:** Replace $\overline{U} \times \overline{V}$ by $\overline{V} \times \overline{U}$.

(2/15/07) **Pages 229–230, proof of Theorem 9.22:** There is a serious gap in the proof that H acts properly on G. Although the sequence $\{h_i\}$ constructed at the top of page 230 converges in G to a point in H, it does not follow that it converges in the topology of H, since we are not assuming that H is embedded. What is needed is the following converse to Proposition 8.30. (This converse is an immediate consequence of Theorem 20.10, the closed subgroup theorem, but that is not available in Chapter 9.)

Proposition. *If G is a Lie group and $H \subset G$ is a closed Lie subgroup, then H is embedded.*

Sketch of Proof. The result is trivial if $\dim H = \dim G$, so we may assume that H has positive codimension in G. It suffices to show that there is some point $h_1 \in H$ and a neighborhood of h_1 in G in which H is embedded, for then composition with left translation yields a slice chart centered at any other point of H. By Lemma 8.18, there exist a neighborhood V of e in H and a slice chart (U, φ) for V in G centered at e. We may assume that $\varphi(U) = B_1 \times B_2$, where B_1 and B_2 are Euclidean balls and $\varphi(V) = B_1 \times \{0\}$. The set $S = \varphi^{-1}(\{0\} \times B_2)$ is an embedded submanifold of G. By the inverse function theorem, the map $\psi\colon V \times S \longrightarrow G$ obtained by restricting group multiplication, $\psi(v, s) = vs$, is a diffeomorphism from a product open set $V_0 \times S_0 \subset V \times S$ to a neighborhood U_0 of e.

Let $K = S_0 \cap H$. Careful analysis of ψ shows that $\psi(V_0 \times K) = H \cap U_0$, and that K is discrete in the topology of H and thus countable. Because H is closed in G, it follows that K is closed in S_0. By an easy application of the Baire category theorem, it follows that a closed, countable subset of a manifold must have an isolated point, so there is a point $h_1 \in K$ that is isolated in S_0. This means there is a neighborhood S_1 of h_1 in S_0 such that $S_1 \cap H = \{h_1\}$, so $U_1 = \psi(V_0 \times S_1)$ is a neighborhood of h_1 in G with the property that $U_1 \cap H$ is a slice.

- **Page 233, lines 1 & 2:** Replace "quotient manifold theorem" by "homogeneous space construction theorem."

(1/16/07) **Page 233, lines 9 & 10:** Change the last sentence of the proof to "By Proposition 7.18, $\widetilde{F}$ is smooth, and by the equivariant rank theorem together with Theorem 7.15, it is a diffeomorphism."

- **Page 233, line 4 from the bottom:** Replace "inverse function theorem" by "constant-rank level set theorem."
- **Page 237, Figure 9.7:** The label $\mathrm{O}(n)$ should be replaced by $\mathrm{SO}(n)$.
- **Page 250, line 7 from the bottom:** The second occurrence of F_{k-1} should be F_k.
- **Page 257, proof of Proposition 10.20, first line:** Change "containing M" to "containing M_0."

- **Page 258, proof of Proposition 10.22, just above the displayed equation:** The range of $\overline{H}$ should be N, not M.

(11/10/06) **Page 263, second line from bottom:** Change "arbitrary" to "appropriate."

- **Page 270, Proposition 11.8(e):** Add the hypothesis that F is a diffeomorphism.
- **Page 270, Exercise 11.8:** Change "Proposition 11.9" to "Proposition 11.8."
- **Page 270, paragraph after Exercise 11.8:** In the first sentence of the paragraph, replace "the category of smooth manifolds and smooth maps" by "the subcategory SM_1 of SM consisting of smooth manifolds and diffeomorphisms."
- **Page 279, last paragraph:** Replace the first three sentences in this paragraph with the following: "Conversely, suppose that W is open in the metric topology, and let $p \in W$. Let $\overline{V}$ be a smooth coordinate ball of radius r around p, let $\overline{g}$ be the Euclidean metric on $\overline{V}$ determined by the given coordinates, and let c, C be positive constants such that (11.6) is satisfied for $X \in T_qM$, $q \in \overline{V}$. Let $\varepsilon < r$ be a positive number small enough that the metric ball around p of radius $C\varepsilon$ is contained in W, and let V_ε be the set of points in $\overline{V}$ whose Euclidean distance from p is less than ε."
- **Page 281, line 3 from the bottom:** Change "Proposition 10.17" to "Lemma 10.17."
- **Page 282, Exercise 11.16:** Add the following sentence at the end of the exercise: "(The same is true when S is merely immersed; the proof is a bit more complicated but straightforward.)"
- **Page 298, line 5:** Change the first "multi-index" to "index."
- **Page 303, third displayed equation:** In this equation and in the line above it, change ω^I to ω_I.
- **Page 305, proof of Corollary 12.13:** Change G to F.
- **Page 307, last line:** The sentence beginning "Properties (a)–(c) ..." should be moved to the end of the next paragraph, after "This completes the proof of the existence and uniqueness of d in this special case." (Until d is proved to be unique, we don't know that it is given by (12.15) in *every* smooth chart.)
- **Page 308, last displayed equation:** Add one extra equality, so that the entire equation reads

$$0 = \widetilde{d}(\varphi\eta)_p = \widetilde{d}\varphi_p \wedge \eta_p + \varphi(p)\widetilde{d}\eta_p = d\varphi_p \wedge \eta_p + \varphi(p)\widetilde{d}\eta_p = \widetilde{d}\eta_p.$$

- **Page 308, line 3 from the bottom:** Replace $d\eta$ by $\widetilde{d}\eta$.
- **Page 311, displayed equation above Exercise 12.6:** The right-hand side should be summed only over $j < k$:

$$d\varepsilon^i = -\sum_{j<k} c^i_{jk}\, \varepsilon^j \wedge \varepsilon^k.$$

- **Page 315, Exercise 12.8(d):** The last part of the sentence should read "and $\dim S = \frac{1}{2}\dim V$."
- **Page 317, line above the first displayed equation:** Replace $T_{(q,\varphi)}(T^*Q)$ by $T^*_{(q,\varphi)}(T^*Q)$.
- **Page 318, Figure 12.4:** The label ${}_qQ$ should be T_qQ.
- **Page 321, First line after the displayed commutative diagram:** Change the beginning of this sentence to "Define a wedge product on $\bigoplus_k A^k(V^*)$ by ...."
- **Page 321, Line 5 after the displayed commutative diagram:** Change $\Lambda^k(V)$ to $\Lambda^*(V)$.
- **Page 323, Problem 12-17:** Add the hypothesis that the forms $\alpha^1, \ldots, \alpha^k$ are smooth.

- **Page 337, proof of Proposition 13.12, line 3 from the bottom:** Change E_n to E_{n-1}.
- **Page 342, last line of Example 13.21:** Replace D by $(0,1] \times U$.
- **Page 347, Problem 13-7:** Insert "M" after "$2n$-manifold" and before the comma.
- **Page 347, Problem 13-9(a):** Add the following sentence at the end of the problem statement: "(You may use without proof the fact that the result of Exercise 11.16 holds also for immersed submanifolds.)"
- **Page 356, line 7:** Change "any n-form" to "any compactly supported n-form."
- **Page 362, first displayed equation:** In the last term, replace b by a.
- **Page 366, two lines above Proposition 14.18:** After the phrase "whose boundary has measure zero in ∂M," insert "and whose closure is compact."
- **Page 382, last line of the top displayed equation:** Replace dV_g by $dV_{\widetilde{g}}$.
- **Page 382, Problem 14-1:** Change the phrase "with the orientation determined by its product structure (see Exercise 13.4)" to "with the orientation determined by $d\theta^1 \wedge d\theta^2$, where θ^1 and θ^2 are angle coordinates on the first and second copies of $\mathbb{S}^1$, respectively."
- **Page 397, proof of Theorem 15.9:** In line 3 of the proof, replace "chain maps" by "cochain maps."
- **Page 408, Problem 15-6:** Add the hypothesis that N is connected.
- **Page 408, Problem 15-7:** Add the hypothesis that M_1 and M_2 are compact.
- **Page 413, line 11 from the bottom:** All three occurrences of F should be changed to $F_\#$, so the sentence begins "An easy computation shows that $F_\# \circ \partial = \partial \circ F_\#$, so $F_\#$ is $\dots$."
- **Page 413, line 8 from the bottom:** Change $(Id_M)_*$ to $(\mathrm{Id}_M)_*$ [Id should not be in italics].
- **Page 422, last paragraph, third line:** Replace $(e_1, 1)$ by $(e_i, 1)$.
- **Page 426, third displayed equation:** Replace ω by η in the third integral.
- **Page 430, line 10:** Replace "for m even" by "for $m \in \mathbb{Z}$."
- **Page 439, proof of Proposition 17.3:** In the first line of the proof, replace "Lemma 4.2" by "Lemma 4.6."
- **Pages 452–453, statement of Lemma 17.16:** Replace "*nonnegative constants A and B*" by "*constants $A > 0$ and $B \geq 0$.*"
- **Pages 456–457, proof of Theorem 17.19:** The inequality on the first line of page 457 is incorrect, because $\theta(s, \widetilde{x})$ might not lie in $\overline{U}_1$ for all s between t and $\widetilde{t}$, so we cannot conclude that $|V(\theta(s, \widetilde{x}))| \leq M$. To fix the proof, make the following changes: (1) In the first sentence of the proof, replace "on some neighborhood of (t_1, x_1)" by "at (t_1, x_1)." (2) Replace the unnumbered displayed equation three lines above (17.19) by
$$M = \sup_{t \in \overline{J}_1} |V(\theta(t, x_1))|, \qquad T = \sup_{t \in \overline{J}_1} |t - t_0|.$$
(3) Just below that same displayed equation, insert the parenthetical remark "(Because $t \mapsto \theta(t, x_1)$ is an integral curve of V, it is continuous and therefore M is finite.)" (4) In the paragraph that starts near the bottom of page 456 and continues onto page 457, replace the first sentence by "Let $(t, x) \in \overline{J}_1 \times \overline{U}_1$ be arbitrary." (5) In the remainder of that same paragraph, replace every instance of $\widetilde{t}$ by t_1 and every instance of $\widetilde{x}$ by x_1. (6) Finally, replace the last sentence of that paragraph (near the top of p. 457) by "It follows that θ is continuous at (t_1, x_1)."

- **Page 457, fifth line below equation (17.21):** Change "nonzero vector $h \in \overline{B}_r(0) \subset \mathbb{R}^n$" to "real number h such that $0 < |h| < r$."

- **Page 468, statement of Lemma 18.4:** Replace the last sentence and the accompanying diagram by the following: "*If X and Y are F-related, then for each $t \in \mathbb{R}$, $F(M_t) \subset N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:*

$$\begin{tikzcd} M_t \arrow[r, "F"] \arrow[d, "\theta_t"'] & N_t \arrow[d, "\eta_t"] \\ M \arrow[r, "F"'] & N. \end{tikzcd}$$

  Conversely, if for each $p \in M$ there is an $\varepsilon > 0$ such that $\eta_t \circ F(p) = F \circ \theta_t(p)$ for all $|t| < \varepsilon$, then X and Y are F-related.

- **Page 469, statement of Proposition 18.5(f):** Replace the statement by the following: "*For each $p \in M$, $\theta_t \circ \psi_s(p) = \psi_s \circ \theta_t(p)$ provided that either $\psi_s \circ \theta_\tau(p)$ is defined for all $\tau \in [0, t]$, or $\theta_t \circ \psi_\sigma(p)$ is defined for all $\sigma \in [0, s]$. In particular, this is true for all s and t if V and W are complete.*"

- **Page 470, last paragraph:** Replace the two sentences beginning with "Thus Lemma 18.4 applied with . . . " by the following: "Thus Lemma 18.4 applied with $F = \psi_s : M_s \longrightarrow M_{-s}$, $X = V|_{M_s}$, and $Y = V|_{M_{-s}}$ implies that $\theta_t \circ \psi_s(p) = \psi_s \circ \theta_t(p)$ provided that $\theta_\tau(p) \in M_s$ for all τ between 0 and t. Since (e) implies (d), the same argument with V and W reversed shows that this also holds when $\psi_\sigma(p)$ is in the domain of θ_t for σ between 0 and s."

- **Page 481, second paragraph after the statement of Theorem 18.19:** Replace the second sentence of this paragraph by the following: "The proof we will give was discovered in 1971 by Alan Weinstein [Wei71], based on a technique due to Jürgen Moser [Mos65]."

- **Page 487, line 18:** Change "electromagnetic" to "electrostatic."

- **Page 488, first displayed equation:** Change v_i to v^i.

- **Page 488, two lines below equation (18.22):** Change T^*M to T^*Q.

- **Page 488, three lines below equation (18.22):** Change $E(p, q)$ to $E(q, p)$.

- **Page 492, Problem 18-9:** Replace the first two sentences in the problem statement by the following: "Prove the following global version of the Darboux theorem, due to Moser [Mos65]: Let M be a compact manifold, and let $\{\omega_t : t \in [0, 1]\}$ be a family of symplectic forms on M. Suppose there exists a smooth function $f : [0, 1] \times M \longrightarrow \mathbb{R}$ such that $\omega_t = \omega_0 + df_t$ for each $t \in [0, 1]$, where $f_t(x) = f(t, x)$."

- **Page 495, line 14 from the bottom:** Change "An immersed submanifold" to "A nonempty immersed submanifold."

- **Page 500, proof of Proposition 19.9, second line:** Change "defined on an open set $U \subset M$" to "that annihilates D on an open set $U \subset M$."

- **Page 505, proof of Proposition 19.13, fourth line from end:** Replace $F : N \longrightarrow H$ by $F : M \longrightarrow H$.

- **Page 507, line 11 from the bottom:** Change $\mathbb{R}^n$ to $\mathbb{R}^3$.

- **Page 517, Problem 19-11:** Change $x > 0$ to $y > 0$ in the definition of U.

- **Page 526, statement of Proposition 20.9:** The range of the function Z should be $\mathfrak{g}$, not $\mathbb{R}$.

- **Page 526, proof of Proposition 20.9:** In the displayed equation on line 8 of the proof, the last G should be $\mathfrak{g}$.

- **Page 544, last line:** Insert "in" before "the following lemma."
- **Page 552, statement of Lemma A.17:** In part (e), replace "*compact subsets*" by "*disjoint compact subsets.*"
- **Page 573, line below equation (A.7):** Change $1 \times n$ to $1 \times (n-1)$.
- **Page 599, just after reference [War83]:** Insert the following reference:
  [Wei71] Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Adv. Math.*, 6:329–346, 1971.

(5/5/06) **Page 602, left column:** Insert the following index entries after $\mathcal{B}_p(M)$:

$B_r(x)$ (open ball in a metric space), 542
$\overline{B}_r(x)$ (closed ball in a metric space), 542

- **Page 623, right column, line 3 from the bottom:** The last index entry for "smooth/structure/ uniqueness of" should refer to page 461, not 460.
- **Page 627, right column, after line 1:** Insert the entry "universal covering manifold, 41."